On the Conway-Maxwell-Poisson Process Model

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Motivation

This work proposed a new Conway-Maxwell-Poisson process (CMPP) to model the replicated spatial point patterns, which are commonly observed in practice).

Methodology

[2] defined a *CMP distribution* by its probability mass function

$$f(i) = \frac{\gamma^i}{Z(\gamma,\nu)(i!)^\nu}, \quad i \in_+ := \{0,1,2,\dots\}, \tag{1}$$

Models

We propose a class of inhomogeneous CMPP(ν, λ) by drawing the number of points according to the pmf in (1) with parameters $\gamma, \nu \geq 0$; and sampling the points independently over W according to the pdf $\lambda(\cdot)/\ell(\lambda)$. The log-likelihood of K independent samples is [3]

$$-(\nu-1)\sum_{k=1}^{K}\log(n_k!) - \sum_{k=1}^{K}\log\big(Z\big(\ell(\lambda^{(k)}),\nu\big)\big) + \sum_{k=1}^{K}\sum_{i=1}^{n_k}\log\big(\lambda^{(k)}(z_i^{(k)})\big). \tag{2}$$

The intractable constants $\log\big(Z\big(\ell(\lambda^{(k)}),\nu\big)\big)$ render it an impractical way to make statistical inference.

Inferences

The method of using logistic regression for making inference for point process [1] hinges on the Papangelou conditional intensity $\pi(z, \{z_1, \dots, z_n\})$ of the point process, which is given by

$$\begin{aligned}&\frac{j(\{z_1,\dots,z_n\}\cup\{z\})}{j(\{z_1,\dots,z_n\}\setminus\{z\})}\\ &= \exp\big(\log(\lambda(z)) + (1-\nu)\log(n + \mathbf{1}_{\{z\notin\{z_1,\dots,z_n\}\}})\big)\\ &= \exp\big(\boldsymbol{\theta}^T \boldsymbol{t}(z,\{z_1,\dots,z_n\}\setminus\{z\})\big),\end{aligned}$$

$\boldsymbol{\theta}^T = (\boldsymbol{\beta}^T, 1-\nu)$ and t is a function.
Parameters can be estimated and inference can be made through a logistic regression using an additional artificial covariate with value $\log(\mathbf{1}_{\{z\notin\{z_1,\dots,z_n\}\}} + n)$. This can be easily conducted using the **spatstat** package in the R programming language.

Simulation Studies

Pattern Correlation

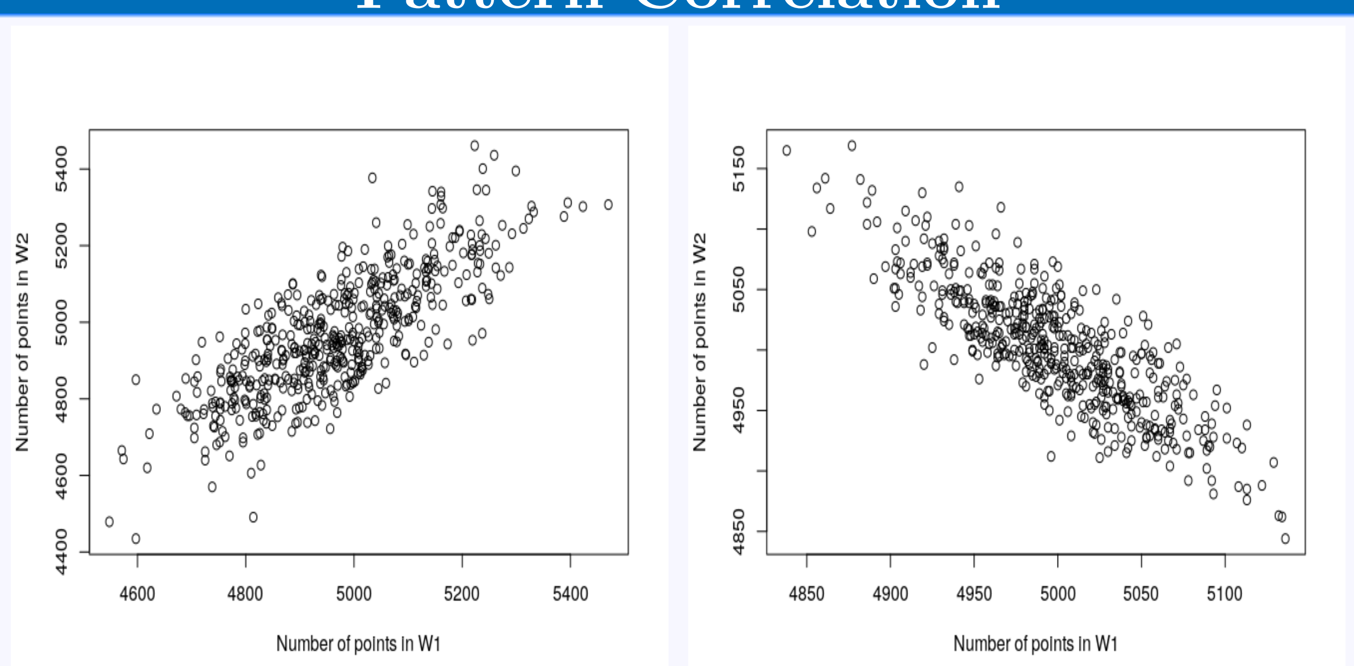

Figure: Scatter plots of the total number of individuals in two subregions W_1 and W_2 for the (λ, ν). The left panel shows the result for the parameters $(2.51, 0.1)$ with positive relation; the right panel corresponds to $(10^{40}, 10)$ with negative relation. The number of replications is 500.

Simulation Study

Inhomogeneous Spatio-temporal Process

We consider a sequence of independent inhomogeneous samples, coming from a $(\lambda^{(k)}, \nu)$, where $\lambda^{(k)}(s)$ depends on a spatio-temporal covariate $x^{(k)}(s)$ at location $s \in W = [0,1]^2$ according to

$$\lambda^{(k)}(s) = \exp(\beta_0 + \beta x^{(k)}(s)). \tag{3}$$

Here

$$x^{(k)}(s) = y(s) + \cos\left(\frac{2\pi k}{50}\right),$$

with $y(s)$ the y-coordinate of location s. The true parameter values are given by $(\beta_0, \nu, \beta) = (1.5, 0.8, 2.5)$. The results are presented in Table 1.

Table: Estimates of parameters for the inhomogeneous study.

Parameter	True value	Mean estimate	Coverage probability
β_0	1.50	1.52	0.95
ν	0.8	0.808	0.95
β	2.50	2.52	0.92

Capeweed Study

We modelled the Capeweed point pattern data (from GBIF) between 2012 and 2018, with quasi-intensity on the k-th year given by

$$\lambda^{(k)}(s) = \exp\left[\beta_0 + \sum_{i=1}^{p}\beta_i X_i^{(k)}(x) + \gamma Y(x)\right], \tag{4}$$

where $X_1^{(k)}, \dots, X_p^{(k)}$ are environmental covariates important in explaining the distribution of the species, and $Y(x)$ is the accessibility to city bias layer from bias.

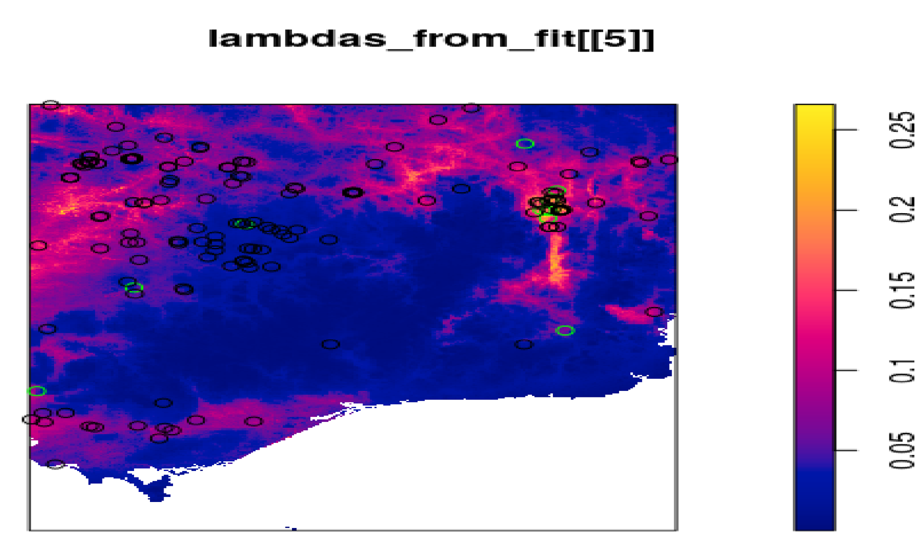

Figure: Sightings of capeweed between 2012 and 2018 in South-East Australia (in black) with those in year 2017 highlighted in green, along with the predicted quasi-intensity in 2017.

References

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