Exploring the performance of *Inductive Linearisation* for simulation of the Van der Pol oscillator

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Background

- The *Inductive Linearisation* solver (1) approximates solutions to nonlinear systems utilising iterative linearisation to create a linear time-varying (LTV) system and is linked to *eigenvalue decomposition* (EVD) for integration.
- This method has been optimised and compared favourably to an inbuilt differential equation solver in MATLAB (*ode45*) for solving non-stiff nonlinear systems in pharmacology (2).
- Here it is explored for solving the Van der Pol system, a nonlinear system of arbitrary stiffness.

Aims

- To evaluate the efficacy of the *Inductive Linearisation* solver when applied to the stiff Van der Pol system.

Methods

- The reference ODE solvers were the MATLAB built-in functions *ode45* (non-stiff) and *ode23s* (stiff).
- The Van der Pol system was examined for two values of the damping parameter ($\mu = 1, 10$), denoting non-stiff and stiff.
- The second-order differential equation of Van der Pol is given:

$$y'' - \mu(1 - y^2)y' + y = 0;$$

Where, $y''$is the second derivative, $y'$is the first derivative and μ is the damping parameter.

- The function is written as a system of linearised ODEs:

$$\frac{dy_1}{dt} = y_2$$

$$\frac{dy_2^{[n]}}{dt} = -y_1 + \mu\left(1 - {y_1}^{[n-1]^2}\right){y_2}^{[n]}$$

- The matrix of coefficients (K):

$$K = \begin{bmatrix} 0 & 1 \\ -1 & \mu\left(1 - {y_1}^{[n-1]^2}\right) \end{bmatrix}$$

- Run time and graphical precision were examined for the *Inductive Linearisation* solver.
- The *Inductive Linearisation* solver was optimised for (as per (2))
  - Convergence criterion ε for the linearisation, and
  - Adaptive step-size for EVD to have the optimal step-size (ss) & α.
- In addition, we can advantage of the repeating cycles of the oscillator by optimising the inductive method for the first cycle and carrying forward the values of y_1 and y_2 from the first cycle to be the plug-in values of y_0 for all future cycles. This yields a single-step linearization process.

Results

- The optimal performance for solving the whole time span (0 – 40 h) was gained by (Table 1):

1) $\varepsilon = 10e - 6$ (convergence criterion), $N_{max} = 20$, $\alpha = 0.001$ (non-stiff).
2) $\varepsilon = 10e - 6$ (convergence criterion), $N_{max} = 60$, $\alpha = 0.03$ (stiff).

- Solving the first cycle of the non-stiff and stiff system (Figures 1 & 2) using the optimised results (from above) and using this to perform a single step linearization yielded improved performance Table 1.

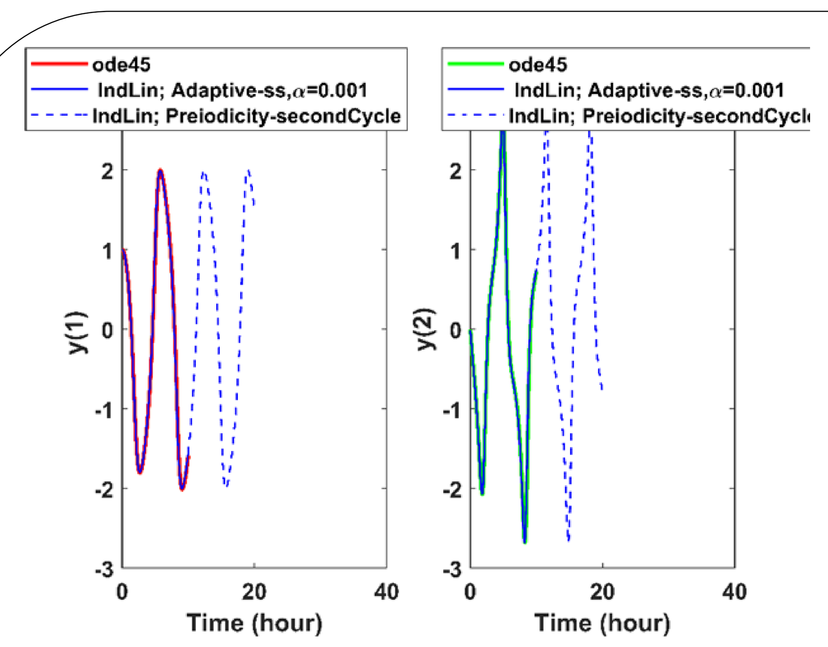

Figure 1: The periodicity for the non-stiff VDP solution using the optimised *Inductive Linearisation* solver where, $\mu = 1$, $ss =$ adaptive step-size using $\frac{dy_2}{dt}$, $\alpha = 0.001$.

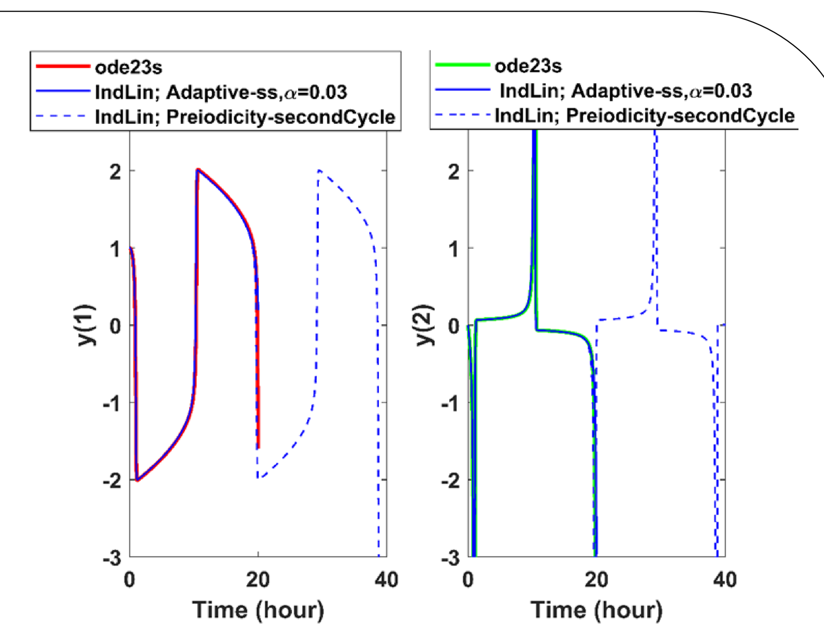

Figure 2: The periodicity for the stiff VDP solution using the optimised *Inductive Linearisation* solver where, $\mu = 10$, $ss =$ adaptive step-size using $\frac{dy_2}{dt}$, $\alpha = 0.03$.

Table 1: The speed of the optimised *Inductive Linearisation* solver to solve VDP

Method	***Run time (s)***
$\mu = 1$	
ode45	0.032
Inductive Linearisation solver for time span	0.572
Inductive linearisation solver for one cycle + one step linearisation for subsequent cycles	0.291
$\mu = 10$	
ode23s	2.47
Inductive Linearisation solver for time span	5.23
Inductive linearisation solver for one cycle + one step linearisation for subsequent cycles	2.07

Conclusions

- The *Inductive Linearisation* solver performed acceptably for both the stiff and non-stiff Van der Pol system.
- Taking advantage of the periodicity yielded significant improvements in performance.

References

1. Duffull SB, Hegarty G, *Theor Comput Sci.* 2014;1(119):2

2. Sharif S, Hasegawa C, Duffull SB. J Pharmacokinet Pharmacodyn. 2022:1-9.